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PROPAGATION OF ELASTIC WAVES THROUGH MEDIA WITH THIN CRACK-LIKE INCLUSIONS*

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Wave propagation in elastic homogeneous media containing a random number of thin inclusions is studied. The material of the inclusions is assumed to be elastic or viscoelastic and appreciably softer than the medium surrounding them. Only the principal terms of the expansion of elastic fields in terms of the small parameters of the problem are considered, namely, the ratio of the characteristic linear dimensions of a typical inclusion and the ratio of the characteristic mdouli of elasticity of the inclusion and the medium. This makes it possible to replace every inclusion by an equivalent singuar model. In the case of statics, analogous models of thin inclusions were given in /1—3/. The model problem of long-wave scattering by a single thin ellipsoidal inclusion is solved explicitly, and the solution is then used to study a medium containing a random number of thin defects. The effective-field method /4, 5/ which takes into account multiple scattering of waves is used to obtain the averaged equation of motion of such a medium (the effective wave operator) in the long-wave approximation. The operator describes the wave propagation in a homogeneous medium with dispersion and attenuation. The velocities of propagation and the attenuation coefficients of various types of elastic waves propagating through materials with randomly oriented inclusions or cracks, and with a system of parallel cracks, are found.

The static moduli of elasticity of media with cracks, and hence the velocities of propagation of long waves in such materials, were determined using the effective field method in /6—8/. Other method were used in /9—11/ to find the attenuation coefficients of elastic waves in a medium with cracks, in the Rayleigh approximation. In the case of a medium with cracks, the results of this paper agree with those obtained in the papers listed above.

1. A model of a thin inclusion in an elastic medium. Let an unbounded homogeneous elastic medium with the tensor of elastic moduli C_{ijkl} and density ρ contain a region V with elastic characteristics C'_{ijkl} and density ρ'. We shall assume that one of the characteristic dimensions of this region, namely h, is small compared with the other two, and that the moduli of elasticity of the inclusion are appreciably smaller than those of the medium. We shall choose, at every point x of the middle surface Ω of the region V, a local coordinate system y_1, y_2, y_3 with the axis y_3 directed along the normal $n(x)$ to the surface Ω. We denote by $h(x)$ the transverse dimension of the region V along the y_3 axis. The function $h(x)$ and tensor C' can be represented in the form

$$h(x) = \delta_1 l(x), \quad C'_{ijkl} = \delta_2 \overset{\circ}{C}_{ijkl} \tag{1.1}$$

where δ_1 and δ_2 are small dimensionless parameters, $l(x)$ is of the order of the largest linear dimension of the region V, and the components of the tensor C° are of the same order as the moduli of elasticity of the basic medium.

Below, we shall assume that $h(x)$ is a fairly smooth function satisfying the condition $|\partial h(x)| \ll 1$ everywhere on Ω, with the exception of a small neighbourhood of the contour Γ of the boundary Ω. Here the symbol ∂ denotes the grad operation along the surface Ω

$$\partial_i = \nabla_i - n_i(x) n_j(x) \nabla_j, \ \nabla_i = \partial/\partial x_i, \quad x \in \Omega \tag{1.2}$$

Let us consider the problem of the propagation of elastic stationary waves of frequency ω through a medium with a thin defect. Using the smallness of the transverse dimension of this defect, we can replace the initial problem by a boundary value problem for a medium with the boundary conditions at the surface Ω, which approximately models the presence of an inclusion. Such boundary conditions were formulated in /1, 2/ in the static case $(\omega = 0)$. In dealing with the problem of the stationary oscillation of a medium with a thin defect, these conditions can be generalized in a natural was as follows. If we denote the amplitude values of the displacement vector and stress tensor by $u(x)$ and $\sigma(x)$, then for $x \in \Omega$ we have

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$$[u_i(x)] = b_i(x), \quad [n_i(x)\sigma_{ij}(x)] = \omega^2 v_j(x) \tag{1.3}$$

$$n_j(x)\bar{\sigma}_{ij}(x) = \frac{1}{h(x)} n_j(x) C'_{ijkl} n_k(x) b_l(x), \quad v_i(x) = h(x)\rho_1 \bar{u}_i(x) \tag{1.4}$$

Here the square brackets denote the difference in the limit values of the function $f(x)$ approaching Ω from the direction of the normal (f^+) and from the opposite direction (f^-) : $[f] = f^+ - f^-$, $b_i(x)$ is the unknown displacement jump vector on Ω, $\rho_1 = \rho' - \rho$, $\bar{u}$ and $\bar{\sigma}$ are the mean values of the displacements and stresses

$$\bar{u}(x) = {}^1/_2(u^+(x) + u^-(x)), \quad \bar{\sigma}(x) = {}^1/_2(\sigma^+(x) + \sigma^-(x)) \tag{1.5}$$

We note that the quantity $\omega^2 v_j(x)$ in (1.3), which can be regarded as the force of inertia, is proportional to the fluctuation in the density ρ_1 of the material in the region occupied by the inclusion. Below, we shall assume that for small δ_1 the inertial term on the right-hand side of the second relation of (1.3) can be neglected $(v_i = 0)$. Here the conditions (1.3) and (1.4) are essentially the same as those in /1, 2/.

We obtain the correct boundary value problem of the dynamic theory of elasticity for a medium with a thin inclusion, by adding to (1.3) and (1.4) the condition at infinity

$$u(x) \to u^\circ(x) \quad \text{as} \quad |x| \to \infty \tag{1.6}$$

where $u^\circ(x)$ is the "incident" field which would exist in the medium without an inhomogeneity, and the manner in which $u(x)$ tends to $u^\circ(x)$ is determined by the well-known radiation conditions.

It can be shown /3/ that the solution of the boundary value problem formulated here yields an asymptotic expression for the wave field outside the inclusion with an accuracy of up to terms of the order of the small parameters δ_1 and δ_2, provided that $\delta_2/\delta_1 = O(1)$.

We shall seek the displacement field $u(x)$, the deformation field $\varepsilon(x)$ and the stress field $\sigma(x$ outside the inclusion in the form of the following potentials:

$$u(x) = u^\circ(x) - u_1(x), \quad u_1(x) = \int_\Omega \nabla g(x - x') C n(x') b(x') d\Omega' \tag{1.7}$$

$$\varepsilon(x) = \varepsilon^\circ(x) + \varepsilon_1(x), \quad \varepsilon_1(x) = \int_\Omega K(x - x') C n(x') b(x') d\Omega'$$

$$\sigma(x) = \sigma^\circ(x) + \sigma_1(x), \quad \sigma_1(x) = \int_\Omega M(x - x') n(x') b(x') d\Omega'$$

Here $g_{ik}(x)$ is Green's tensor of the wave operator for the medium, satisfying the equation

$$(L_{ik}g_{kj})(x) = -\delta(x)\delta_{ij}, \quad L_{ik} = \nabla_j C_{ijkl} \nabla_l + \rho\omega^2 \delta_{ik} \tag{1.8}$$

and the kernels of the potentials $\varepsilon_1(x)$ and $\sigma_1(x)$ are connected with the second derivatives of Green's tensor $g_{ik}(x)$ by the relations

$$K_{ijkl}(x) = -g_{i)(k,l)(j}(x) \tag{1.9}$$
$$M_{ijkl}(x) = C_{ijmn} K_{mnrs}(x) C_{rskl} - C_{ijkl}\delta(x)$$

When the anisotropy of the medium is arbitrary, Green's tensor $g_{ik}(x)$ can be represented in the form of the following series /5, 9/:

$$g(x) = g_0(x) + \frac{1}{|x|} \sum_{k=1}^{\infty} \frac{(i\omega|x|)^k}{(k-1)!} g_k(n) \tag{1.10}$$

$$g_0(x) = \frac{1}{8\pi^2 |x|} \int_{|\xi|=1} \Lambda^{-1}(\xi)\delta(n\cdot\xi) dS_\xi$$

$$g_k(n) = \frac{1}{16\pi^3} \int_{|\xi|=1} \Lambda^{-{}^1/_2(k+2)}(\xi) |n\cdot\xi|^{k-1} dS_\xi$$

$$\Lambda_{ik}(\xi) = \rho^{-1}\xi_j C_{ijkl}\xi_l, \quad n = x/|x|$$

(ξ_i is a vector on the surface of the unit sphere). We note that $g_0(x)$ represents a "static" Green's tensor, i.e. Green's tensor of the operator L_{ij} when $\omega = 0$.

When the solution is chosen in the form (1.7), the field $u(x)$ satisfies the homogeneous wave equation $L_{ik}u_k(x) = 0$, the conditions (1.6) and the radiation conditions. Outside Ω the potentials u, ε and σ are connected by the relations $\varepsilon = \operatorname{def} u$, $\sigma = C\varepsilon$. At the points of the surface Ω the functions $\varepsilon_1(x)$ and $\sigma_1(x)$ (1.7) are discontinuous, and it will be shown below that the jumps in the values of these potentials satisfy the conditions (1.3) automatically for any $b(x)$ and $v(x) = 0$. The remaining condition (1.4) enables us to determine the function $b(x)$ uniquely. Next we shall derive an equation which must be satisfied by the function $b(x)$.

2. The integral equation of the problem.

First we shall study the limit values

of the potentials $u_1(x)$, $\varepsilon_1(x)$ and $\sigma_1(x)$ as $x \to \Omega$. In accordance with the expansion of Green's tensor in the series (1.10), the functions $K(x)$ and $M(x)$ can also be written in the form of the sum of the "static" and "dynamic" components

$$K(x) = K^\circ(x) + K^\omega(x),\ M(x) = M^\circ(x) + M^\omega(x) \tag{2.1}$$

We note that the static parts of these functions $K^\circ(x)$ and $M^\circ(x)$ are homogeneous generalized functions of degree -3, whose regularization was given in /4/. The functions $K^\omega(x)$ and $M^\omega(x)$ are integrable on Ω, since they have a singularity $|x|^{-1}$ at zero.

The potentials $u_1(x)$, $\varepsilon_1(x)$ and $\sigma_1(x)$ can also be written, by virute of (2.1), in the form of a sum of the static and dynamic components. Since the dynamic parts are continuous on Ω, it follows that only the static parts undergo a jump when passing through Ω.

Let us first consider the potential $u_1(x)$. This integral represents the potential of the double layer of the dynamic theory of elasticity. Its limit values on Ω are given by the relations /12/

$$u_1^\pm(x) = \int_\Omega \nabla g(x-x') Cn(x') b(x') d\Omega' \pm b(x) \tag{2.2}$$

(the integral is regarded as the princpal Cauchy value).

Let us now consider the potential $\sigma_1(x)$. It can be shown that the vector $n_j(x)\sigma_{ij}^1(x)$ is continuous on passing through the surface Ω, and its value on Ω is obtained from the following regularization /13/:

$$\begin{aligned}
-n_j(x)\sigma_{ij}^1(x) &= \int_\Omega T_{ij}(x,x') b_j(x') d\Omega' = \\
&\int_\Omega T_{ij}^\circ(x,x')[b_j(x') - b_j(x)] d\Omega' + \Gamma_{ij}(x) b_j(x) + \\
&\int_\Omega T_{ij}^\omega(x,x') b_j(x') d\Omega', \quad x \in \Omega \\
T_{ij}(x,x') &= T_{ij}^\circ(x,x') + T_{ij}^\omega(x,x') \\
T_{ij}^\circ(x,x') &= -n_k(x) \overset{\circ}{M}_{ikjl}(x-x') n_l(x'), \quad T_{ij}^\omega(x,x') = \\
&-n_k(x) M_{ikjl}^\omega(x-x') n_l(x')
\end{aligned} \tag{2.3}$$

where $\Gamma_{ij}(x)$ is the contour integral along the boundary of Ω, whose explicit form was given in /13/.

From relation (2.2) and the continuity of the vector $n(x)\sigma_1(x)$ on Ω it follows that when $v_i = 0$, conditions (1.3) for $u(x)$ and $\sigma(x)$ in the form (1.7) are satisfied automatically and conditions (1.4) together with (1.5) lead to the following expression for the unknown function $b(x)$:

$$\frac{1}{h(x)} n_j(x) C'_{ijkl} n_l(x) b_k(x) + \int_\Omega T_{ik}(x,x') b_k(x') d\Omega' = n_k(x)\sigma_{ik}^\circ(x) \tag{2.4}$$

in which the integral with the kernel $T_{ik}(x,x')$ should be regarded as the right-hand side of (2.3). Finding $b(x)$ from the above expression and substituting the result into (1.7), we obtain the solution of the problem in question. In the case of thin ellipsoidal inclusions and wavelengths appreciably exceeding the maximum dimensions of the inclusion, the problem can be solved by quadratures. We shall consider this case in more detail.

3. Scattering of long waves by a thin ellipsoidal defect. Let $l/\lambda \ll 1$ where λ is the wavelength of the incident wave. Since $x\omega \sim x/\lambda$, it follows that in solving the integral Eq.(2.4) we can restrict ourselves to the first terms of the expansion of the dynamic Green's tensor in the series (1.10) in powers of ωx. Moreover, we will neglect, in the real part of the Green's tensor, terms of the order of $(\omega x)^2$ and higher as compared with the static part $g_0(x)$, and in the imaginary part we shall retain terms of the order of up to and including $(\omega x)^3$. This will enable us to determine correctly the terms of the real and imaginary parts of the field scattered by the inhomogeneity /5/ that are principal in ω. The expression for the dynamic Green's tensor, with the accuracy up to the terms shown, takes the form

$$g_{ik}(x) = g_{ik}^\circ(x) + i\omega g_{ik}^1 - i\omega^3 x^2 g_{ik}^3(n) \tag{3.1}$$

while Eq.(2.4) becomes

$$\begin{aligned}
&t_{ik}(x) b_k(x) + \int_\Omega [\overset{\circ}{T}_{ik}(x,x') - i\omega^3 T_{ik}^\omega(x,x')] b_k(x') d\Omega' = n_k(x)\overset{\circ}{\sigma}_{ik}(x) \\
&t_{ik}(x) = \frac{1}{h(x)} n_j(x) C'_{ijkl} n_l(x), \quad T^\omega(x,x') = n(x) CHCn(x') \\
&H = (H_{ijkl}) = \frac{1}{16\pi^3} \int_{|\xi|=1} \xi_i \Lambda_{ik}^{-1/2}(\xi)\, \xi_l\, dS_\xi
\end{aligned} \tag{3.2}$$

For a thin ellipsoidal inhomogeneity the surface Ω is an ellipse with semi-axes a_1, a_2, and the function $h(x)$ has the following form in the coordinate system whose axes coincide with the principal axes of the ellipsoid:

$$h(x)=hz(x), \quad z(x)=\sqrt{1-\left(\frac{x_1}{a_1}\right)^2-\left(\frac{x_2}{a_2}\right)^2}, \ \frac{h}{a_1}, \frac{h}{a_2} \ll 1 \tag{3.3}$$

For sufficiently long wavelengths the external field $u_i^\circ(x)$ in the region Ω can be assumed linear

$$u_i^\circ(x) = u_i^\circ + u_{ij}^\circ x_j \ (x \in \Omega) \tag{3.4}$$

Then the deformation tensor ε° and stress tensor σ° are constant for $x \in \Omega$. Moreover, in the case of a flat region the vector n, and hence the tensor T^ω, is also independent of x. As we know /14/, in the case of an elliptical region Ω an operator with the kernel $T^\circ(x, x')$ transforms the function $z(x)$ into a constant. Therefore we can seek a solution of (3.2) in the form

$$b_i(x) = b_i z(x) \tag{3.5}$$

Substituting this expression into (3.2) and restricting ourselves to terms of the order of ω^3, we find the constant vector b

$$b=b^\circ+i\omega^3 b^\omega, \quad b^\circ=\frac{2a_1^2}{a_2} An\sigma^\circ, \ A=\frac{a_2}{2a_1^3}\left(\frac{1}{h}nC'n+T^\circ\right)^{-1} \tag{3.6}$$

$$T^\circ=\int T^\circ(x)\left[z(x)-1\right]d\Omega, \quad b^\omega=\frac{2a_1^2}{a_2} vAT^\omega An\sigma^\circ, \quad v=\frac{4}{3}\pi a_1^3$$

Here the integral T° is computed over the whole plane x_1, x_2, and the function $z(x)$ is continued to zero outside the region Ω /13/.

In accordance with formulas (1.7), the displacement fields $u(x)$ and $\varepsilon(x)$ outside the thin ellipsoidal defect are given, in the long wave approximation, by the expressions

$$u(x)=u^\circ(x)-\int_\Omega \nabla g(x-x')\Lambda(a_1, a_2)Z(x', a_1, a_2)\varepsilon^\circ(x')d\Omega' \tag{3.7}$$

$$\varepsilon(x)=\varepsilon^\circ(x)+\int_\Omega K(x-x')\Lambda(a_1, a_2)Z(x', a_1, a_2)\varepsilon^\circ(x')d\Omega'$$

$$\Lambda(a_1, a_2)=\Lambda^\circ+i\omega^3\Lambda^\omega, \quad \Lambda^\circ=CnAnC$$

$$\Lambda^\omega=v\Lambda^\circ H\Lambda^\circ, \quad Z(x, a_1, a_2)=\frac{2a_1^2}{a_2}z(x)$$

We note that the case $C'=0$ corresponds to a thin cavity (a crack) and in the case of an inclusion composed of a viscoelastic material, C' is a tensor with complex components.

4. Wave propagation in a medium with a random set of thin ellipsoidal defects.

Let us consider an unbounded medium containing a spatially homogeneous random set of ellipsoidal defects. Let $\Omega_k(x)$ be a delta function concentrated on the middle surface Ω_k of the k-th defect. We shall denote the delta function concentrated on the set $\Omega=\bigcup_k \Omega_k$ of these regions by $\Omega(x)$. In the case of harmonic oscillations, the amplitude of the deformation field $\varepsilon(x)$ in a medium with defects can be written in the same form as (1.7)

$$\varepsilon(x)=\varepsilon^\circ(x)+\int K(x-x')Cn(x')b(x')\Omega(x')dx' \tag{4.1}$$

where $n(x)$ and $b(x)$ are arbitrary vector functions identical with the vector of the normal $n_{(k)}(x)$ and the displacement jump vector $b_{(k)}(x)$ on the surface Ω_k ($k=1, 2, \ldots$).

Let us introduce, for any defect denoted by the number k, a local outer field $\varepsilon_{(k)}^*(x)$ containing this defect. The field $\varepsilon_{(k)}^*(x)$ is defined in the region Ω_k and consists of the outer field $\varepsilon^\circ(x)$, and the fields scattered by all remaining defects. Let us denote by $\varepsilon^*(x)$ the deformation field defined in the region Ω and coinciding with $\varepsilon_{(k)}^*(x)$ when $x \in \Omega_k$. As follows from (4.1), the field can be written in the form

$$\varepsilon^*(x)=\varepsilon^\circ(x)+\int K(x-x')Cn(x')b(x')\Omega(x; x')dx', \ x \in \Omega \tag{4.2}$$

Here $\Omega(x; x')$ denotes the delta function concentrated on the set Ω_x defined as follows: $\Omega_x=\bigcup_{i\neq k}\Omega_i$ for $x \in \Omega_k$.

We shall assume, in accordance with the effective field method /4/, that the field $\varepsilon^*(x)$ is constant in each region Ω_k, but changes, generally speaking, from one region to another. Then from (3.7) it follows that the displacement field $u(x)$ can be expressed in terms of $\varepsilon^*(x)$ by the formula

$$u(x) = u^\circ(x) - \int \nabla g(x - x')\Lambda(x')Z(x')\Omega(x')\varepsilon^*(x')\,dx' \tag{4.3}$$

and the field $\varepsilon^*(x)$ satisfies the equation $(x \in \Omega)$

$$\varepsilon^*(x) = \varepsilon^\circ(x) + \int K(x - x')\Lambda(x')Z(x')\Omega(x;x')\varepsilon^*(x')\,dx' \tag{4.4}$$

Here $Z(x)$ and $\Lambda(x)$ are functions of the form (3.7) on every surface Ω_k.

Let us average Eq.(4.4) over the ensemble of samples of the random set of defects under the condition that $x \in \Omega$. Assuming that the value of the random function $\varepsilon^*(x)$ does not depend statistically on the properties and geometrical characteristics of the defect within which the point x lies, we obtain $(\langle\varepsilon^*(x)|x\rangle = \mathrm{E}^*(x))$

$$\mathrm{E}^*(x) = \varepsilon^\circ(x) + \int K(x - x')\Psi(x,x')\langle\varepsilon^*(x')\,|\,x',x\rangle\,dx' \tag{4.5}$$
$$\mathrm{E}^*(x) = \langle\varepsilon^*(x)\,|\,x\rangle, \quad \Psi(x,x') = \langle\Lambda(x')Z(x')\Omega(x;x')\rangle$$

Here the symbol $\langle\cdot\,|\,x,x',\ldots\rangle$ denotes the averaging operation provided that $x, x', \ldots \in \Omega$. To obtain the closed equation for $\mathrm{E}^*(x)$ we use the "quasicrystalline" approximation /15/

$$\langle\varepsilon^*(x)\,|\,x',\,x\rangle = \langle\varepsilon^*(x)\,|\,x\rangle = \mathrm{E}^*(x) \tag{4.6}$$

The physical meaning of this approximation was discussed in /4, 15/, and Eq.(4.5) takes the form

$$\mathrm{E}^*(x) = \varepsilon^\circ(x) + \int K(x - x')\Psi(x,x')\mathrm{E}^*(x')\,dx' \tag{4.7}$$

In the case of a homogeneous random set of defects, the function $\Psi(x,x')$ depends only on the difference between the arguments. Eq.(4.7) becomes here an equation in convolutions, and its solution in the k-representation has the form

$$\mathrm{E}^*(k) = (I - K_\Psi(k))^{-1}\varepsilon^\circ(k), \quad I = (I_{ijkl}) = \delta_{i(k}\delta_{l)j} \tag{4.8}$$

Here $K_\Psi(k)$ is the Fourier transform of the product $K(x)\Psi(x)$. The Fourier transforms of the functions use the same notation, with the argument changed from to x to k.

Let us now average Eq.(4.3) over the ensemble of samples of the random set of defects. Passing now to the k-representation and using (4.8), we express $u^\circ(k)$ by the Fourier transform of the mean displacement field $U_i(k) = \langle u_i(k)\rangle$

$$u_i^\circ(k) = U_i(k) + k_j g_{ik}(k)\,\Lambda_{kjrs}\Pi_{rsmn}(k)\,k_m U_n(k) \tag{4.9}$$
$$\Pi(k) = (I + K_\Phi(k))^{-1}, \quad K_\Phi(k) = \int K(x)\Phi(x)\exp(-ik\cdot x)\,dx$$
$$\Phi(x - x') = \langle\Lambda(x')Z(x')(\Omega(x') - \Omega(x;x'))\rangle$$
$$\Lambda = \langle\Lambda(x)Z(x)\Omega(x)\rangle$$

We shall assume that the random functions in question are ergodic and that the orientation of the inclusions, their size and the position of their centres within the space are random, statistically independent quantities. Then we have

$$\Phi(x) = \left\langle\frac{v}{v_0}\Lambda(a_1,a_2)f(a_1,a_2,x)\right\rangle, \quad \Lambda = \left\langle\frac{v}{v_0}\Lambda(a_1,a_2)\right\rangle \tag{4.10}$$
$$f(a_1,a_2,x) = 1 - \frac{v_0}{v}\lim_{V\to\infty}\frac{1}{V}\int\limits_V Z(x - x')\Omega(x';x - x')\,dx'$$

where v_0 is the mean volume per defect and the value of the averages on the right-hand sides of these formulas is determined by the distribution of the random semi-axes of the defects and their random orientations.

We note that $f(a_1,a_2,x)$ is a smooth function tending rapidly to zero outside the region with the linear dimension r of the order of the mean distance between the centres of the inhomogeneities. For sufficiently long waves ($|k\cdot x| \ll 1$ when $|x| < r$) we can assume that $\exp(-ik\cdot x) \approx 1$ in the integrand of (4.9). Here the integral K_Φ has a constant value and the tensor Π takes the form

$$\Pi = \Pi^\circ[I - i\omega^3(P\Lambda^\omega + JH\Lambda^\circ)\Pi^\circ], \quad \Pi^\circ = (I + P\Lambda^\circ)^{-1} \tag{4.11}$$
$$\Lambda^\circ = \left\langle\frac{v}{v_0}\Lambda^\circ(a_1,a_2)\right\rangle, \quad \Lambda^\omega = \left\langle\frac{v}{v_0}\Lambda^\omega(a_1,a_2)\right\rangle$$
$$P = \int K^\circ(x)f(x)\,dx, \quad J = \int f(x)\,dx, \quad f(x) \equiv f(x,a_1,a_2)$$

Let us pass in (4.9) to the x-representation and apply to both sides of the expression obtained the operator L_{ik} defined in (1.8). Taking into account the equation $L_{ik}{}^*U_k(x) = 0$, we find that the mean displacement field in the medium with defects satisfies the equation

$$L^*_{ik}U_k(x) = 0, \quad L^*_{ik} = \nabla_j C^*_{ijkl}\nabla_l + \rho\omega^2\delta_{ik} \tag{4.12}$$

Here C^*_{ijkl} is the tensor of effective dynamic elastic moduli given by the relations

$$C^* = C^s - i\omega^3 C^\omega, \quad C^s = C - C^\circ, \quad C^\circ = \left\langle \frac{v}{v_0} C^\circ(a_1, a_2)\right\rangle \tag{4.13}$$

$$C^\circ(a_1, a_2) = \Lambda^\circ(a_1, a_2)\Pi^\circ,$$

$$C^\omega = \left\langle \frac{v^2}{v_0} C^\circ(a_1, a_2) H C^\circ(a_1, a_2)\right\rangle - JC^\circ HC^\circ$$

Thus the mean wave field in a medium with defects satisfies an equation which is formally identical with the equation of motion of a homogeneous medium of density ρ and to the tensor of elastic moduli C^*. Since C^* is a complex quantity, it follows that elastic waves decay in such a medium. The decay is connected with the geometrical scattering at the inhomogeneities. We shall now construct the tensor C^* for particular stochastic models of the set of inclusions in an isotropic medium.

5. A random set of thin defects in an isotropic medium. Let us assume that the materials of the basic medium and of the inclusions are isotropic, with Lamé coefficients λ, μ and λ', μ' respectively. In this case expression (3.6) for the tensor A takes the form

$$A_{ij} = A_1 e_i^1 e_j^1 + A_2 e_i^2 e_j^2 + A_3 n_i n_j \tag{5.1}$$

$$A_1 = \frac{a_2}{2a_1^2}\left(\frac{2\mu'}{h} + T_1^\circ\right)^{-1}, \quad A_2 = \frac{a_2}{2a_1^2}\left(\frac{2\mu'}{h} + T_2^\circ\right)^{-1},$$

$$A_3 = \frac{a_2}{2a_1^2}\left(\frac{2\lambda' + 4\mu'}{h} + T_3^\circ\right)^{-1}$$

$$T_1^\circ = \frac{\mu a_2}{2a_1^2(1-\nu)}[c_1 + \nu(c_2 - 2c_1)]$$

$$T_2^\circ = \frac{\mu a_2}{2a_1^2(1-\nu)}[c_1 + \nu(c_3 - 2c_1)]$$

$$T_3^\circ = \frac{\mu a_2 c_1}{2a_1^2(1-\nu)}, \quad c_1 = \frac{E(m)}{1-m^2}, \quad c_2 = c_1 - \frac{1}{m^2}(E(m) - K(m))$$

$$c_3 = 3c_1 - c_2, \; m = 1 - (a_2/a_1)^2 \; (a_1 > a_2)$$

Here e^1, e^2 are unit vectors of the principal axes of the ellipse whose orientation is determined by the normal n, $K(m)$ and $E(m)$ are total elliptic integrals of first and second kind respectively.

The tensor H_{ijkl} in this case becomes isotropic

$$H_{ijkl} = H_1 E^1_{ijkl} + H_2 E^2_{ijkl}, \quad E^1_{ijkl} = \delta_{ij}\delta_{kl}, \quad E^2 = I - \frac{1}{3}E^1 \tag{5.2}$$

$$H_1 = \frac{\eta^5}{36\pi\rho v_T^5}, \quad H_2 = \frac{3 + 2\eta^5}{60\pi\rho v_T^5}$$

$$\eta = \frac{v_T}{v_L} = \left(\frac{\mu}{\lambda + 2\mu}\right)^{1/2}, \quad v_T = \sqrt{\frac{\mu}{\rho}}, \quad v_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

where v_T, v_L are the rates of propgation of the transverse and longitudinal waves in the medium, and ν is Poisson's ratio of this medium.

Let the centres of defects form a statistically homogeneous and isotropic random field, and let their distribution over the orientations be uniform. We shall assume, in addition, that a spherical neighbourhood of each crack exists where the probability of finding the centres of other cracks within this sphere is low (a model with a constraint imposed on the intersection of cracks /8/). Then the function $f(x)$ is spherically symmetrical and the integral P in (4.11) has the form

$$P = P_1 E^1 + P_2 E^2, \quad P_1 = \eta^2/(9\mu), \quad P_2 = (3 + 2\eta^2)/(15\mu) \tag{5.3}$$

Substituting formulas (5.1)-(5.4) into (4.13), we obtain the following expression for the effective dynamic elastic moduli of the medium with inclusions:

$$C^* = K^* E^1 + 2\mu^* E^2 \tag{5.4}$$

Here

$$K^* = K_s - \frac{i\omega^3 K_\omega}{\pi\rho v_T^5}, \quad K_s = K - \left\langle \frac{v}{v_0} K_A(a_1, a_2)\right\rangle \Pi_1 \tag{5.5}$$

$$\mu^* = \mu_s - \frac{i\omega^3 \mu_\omega}{\pi\rho v_T^5}, \quad \mu_s = \mu - \left\langle \frac{v}{v_0}\mu_A(a_1, a_2)\right\rangle \Pi_2$$

$$K_A(a_1, a_2) = \left(\frac{1}{\eta^2} - \frac{4}{3}\right)^2 \mu^2 A_3$$

$$\mu_A(a_1, a_2) = \frac{\mu^2}{15}[4A_3 + 3(1 + \xi)A_1]$$

$$\Pi_1=\left[1+9P_1\left\langle\frac{v}{v_0}K_A(a_1,a_2)\right\rangle\right]^{-1}$$
$$\Pi_2=\left[1+4P_2\left\langle\frac{v}{v_0}\mu_A(a_1,a_2)\right\rangle\right]^{-1}$$
$$K_\omega=\left\langle\frac{v^2}{v_0}K_\omega(a_1,a_2)\right\rangle-\frac{1}{4}J\eta^5\left\langle\frac{v}{v_0}K_A(a_1,a_2)\right\rangle^2\Pi_1^2$$
$$\mu_\omega=\left\langle\frac{v^2}{v_0}\mu_\omega(a_1,a_2)\right\rangle-\frac{1}{30}J(3+2\eta^5)\left\langle\frac{v}{v_0}\mu_A(a_1,a_2)\right\rangle^2\Pi_2^2$$
$$K_\omega(a_1,a_2)=\mu^2A_3K_Ah_0$$
$$h_0=\frac{1}{9}\left[\frac{1}{4}(3-4\eta^2)\eta\Pi_1^2+\frac{2}{5}(3+2\eta^5)\Pi_2^2\right]$$
$$\mu_\omega(a_1,a_2)=\mu^4\left[\frac{4}{15}A_3^2h_0+\frac{1}{40}A_1^2(1+\xi^2)h_1\right]$$
$$h_1=\frac{4}{5}\left(1+\frac{2}{3}\eta^5\right)\Pi_2^2,\quad \xi=\frac{A_2}{A_1},\quad K=\lambda+\frac{2}{3}\mu$$

Thus the inhomogeneous medium is macroisotropic and the wave Eq.(4.12) separates, in the k-representation, into two independent equations for the longitudinal and transverse waves /5/. The dispersion relations corresponding to each of these waves have the form

$$k^2\varkappa^*-\rho\omega^2=0,\quad k^2\mu^*-\rho\omega^2=0$$
$$\varkappa^*=\varkappa_s-\frac{i\omega^3\varkappa_\omega}{\pi\rho v_T^5},\quad \varkappa_s=K_s+\frac{4}{3}\mu_s,\quad \varkappa_\omega=K_\omega+\frac{4}{3}\mu_\omega$$

This yields the following expressions for the wave numbers:

$$k_L=\frac{\omega}{v_L^*}+i\gamma_L(\omega),\quad k_T=\frac{\omega}{v_T^*}+i\gamma_T(\omega);$$
$$v_L^*=\sqrt{\frac{\varkappa_s}{\rho}},\quad v_T^*=\sqrt{\frac{\mu_s}{\rho}}$$

Here v_L^* and v_T^* are the rates of propagation of the longitudinal and transverse waves in a medium with defects, while γ_L and γ_T are the attenuation coefficients of the corresponding waves per unit length

$$\gamma_L(\omega)=\frac{1}{2\pi}\left(\frac{\omega}{v_L^*}\right)^4\frac{v_L^*\varkappa_\omega}{v_L\eta\mu^2},\quad \gamma_T(\omega)=\frac{1}{2\pi}\left(\frac{\omega}{v_T^*}\right)^4\frac{v_T^*\mu_\omega}{v_T\mu^2} \tag{5.6}$$

We see from the above relations that the velocities v_L^* and v_T^* are independent of the frequency, i.e. there is no dispersion of the velocity within the approximation used. This results from the approximation (3.1) of the expression for the dynamic Green's tensor. The attenuation coefficients γ_L and γ_T in (5.6) are proportional to ω^4, therefore they characterise the Rayleigh wave dispersion in the inhomogeneous medium in question.

If the concentration of the defects is low ($v/v_0 \ll 1$), we can neglect the interaction between them (i.e. the effects of multiple scattering) and the formulas obtained become

$$(v_L^*)^2=\frac{K+\frac{4}{3}\mu}{\rho}\left\{1-\mu\left\langle\frac{v}{v_0}\left[\frac{A_3}{\eta^2}\left(1-\frac{8}{3}\eta^2+\frac{32}{15}\eta^4\right)+\frac{4\eta^5}{15}A_1(1+\xi)\right]\right\rangle\right\} \tag{5.7}$$
$$(v_T^*)^2=\frac{\mu}{\rho}\left\{1-\frac{1}{15}\mu\left\langle\frac{v}{v_0}[4A_3+3A_1(1+\xi)]\right\rangle\right\}$$
$$\gamma_L=\frac{1}{2\pi}\left(\frac{\omega}{v_L}\right)^4\mu^2\left\langle\frac{v^2}{v_0}\left[\frac{A_3^2}{\eta^5}\left(1-\frac{8}{3}\eta^2+\frac{32}{15}\eta^4\right)h_0(\eta)+\frac{16}{15\eta}A_1^2(1+\xi^2)h_1(\eta)\right]\right\rangle,$$
$$\gamma_T=\frac{1}{2\pi}\left(\frac{\omega}{v_T}\right)^4\mu^2\left\langle\frac{v^2}{v_0}\left[\frac{4}{15}A_3^2h_0(\eta)+\frac{1}{40}A_1^2(1+\xi^2)h_1(\eta)\right]\right\rangle$$

Here the quantities $h_0(\eta)$ and $h_1(\eta)$ are given by the formulas (5.5) in which we must put $\Pi_1=\Pi_2=1$.

In the case of thin circular cavities ($\lambda'=\mu'=0$), the formulas become identical with those obtained in /11/ by another method.

We note that in the case of cavities filled with a viscous fluid the wave attenuation is caused not only by geometrical scattering, but also by the absorption properties of the fluid. Here the attenuation coefficients acquire additional terms which are proportional, at low frequencies, to ω^2 and hence represent the principal terms of the expansion of the attenuation coefficients in series in ω.

In conclusion we shall consider the case when all the defects have the same orientation. We shall restrict ourselves, for simplicity, to the case of thin circular cavities (cracks). We shall assume that the cracks form a random Poisson field, i.e. the positions of their centres are statistically independent and the centres themselves are distributed uniformly within the space /8/. Here the tensor C^* has a transversally isotropic symmetry, and its important components in the basis of the coordinate axes $x_1x_2x_3$ with the axis x_3, perpendicular to the planes of the cracks, have the form

$$C^*_{1111} = C^*_{2222} = \lambda + 2\mu - \frac{\langle v\rangle}{v_0}\alpha\mu(1-2\eta^2)^2 - i\omega^3\frac{\langle v^2\rangle}{v_0}(1-2\eta^2)^2 C_1^{\omega}$$

$$C^*_{1122} = C^*_{1111} - 2\mu,\quad C^*_{3333} = \lambda + 2\mu - \frac{\langle v\rangle}{v_0}\alpha\mu - i\omega^3\frac{\langle v^2\rangle}{v_0}C_1^{\omega}$$

$$C^*_{1133} = C^*_{2233} = \lambda - \frac{\langle v\rangle}{v_0}\alpha\mu(1-2\eta^2) - i\omega^3\frac{\langle v^2\rangle}{v_0}(1-2\eta^2)C_1^{\omega}$$

$$C^*_{1212} = \mu,\quad C^*_{1313} = C^*_{2323} = \mu\left(1 - \frac{\langle v\rangle}{v_0}\beta\right) - i\omega^3\frac{\langle v^2\rangle}{v_0}C_2^{\omega}$$

$$\alpha = \eta^{-2}\left[\pi\eta^2(1-\eta^2) + \frac{\langle v\rangle}{v_0}\right]^{-1},\quad \beta = \left[\frac{\pi}{4}(3-2\eta^2) + \frac{\langle v\rangle}{v_0}\right]^{-1}$$

$$C_1^{\omega} = \frac{\mu^2\alpha^2\eta^4 h_0(\eta)}{\pi\rho v_T^5},\quad C_2^{\omega} = \frac{\mu^2\beta^2 h_1(\eta)}{8\pi\rho v_T^5}$$

Let the wave normal be parallel to the axis of symmetry x_3 of the material. Then from the dispersion relation

$$\det(k_j C^*_{ijkl}k_l - \rho\omega^2\delta_{ik}) = 0 \tag{5.8}$$

it follows that the longitudinal and transverse waves can propagate in this direction at velocities v_{L3}^* and v_{T3}^* respectively

$$v^{*2}_{L3} = \frac{1}{\rho}\left(\lambda + 2\mu - \frac{\langle v\rangle}{v_0}\alpha\mu\right),\quad v^{*2}_{T3} = \frac{\mu}{\rho}\left(1 - \frac{\langle v\rangle}{v_0}\beta\right)$$

and the attenuation coefficients γ_{L3} and γ_{T3} are

$$\gamma_{L3} = \frac{1}{2\pi}\frac{\langle v^2\rangle}{v_0}\left(\frac{\omega}{v^*_{L3}}\right)^4\frac{v^*_{L3}}{v_L}\eta^4\alpha^2 h_0(\eta)$$

$$\gamma_{T3} = \frac{1}{2\pi}\frac{\langle v^2\rangle}{v_0}\left(\frac{\omega}{v^*_{T3}}\right)^4\frac{v^*_{T3}}{v_T}\beta^2 h_1(\eta)$$

Let us now assume that the direction of the wave normal is perpendicular to the x_3 axis. From (5.8) it follows that in this case a longitudinal wave may propagate in any direction perpendicular to the axis of symmetry at the velocity

$$(v^*_{L1})^2 = \frac{1}{\rho}\left[\lambda + 2\mu - \frac{\langle v\rangle}{v_0}\alpha\mu(1-2\eta^2)^2\right]$$

with the attenuation coefficient

$$\gamma_{L1} = \frac{1}{2\pi}\frac{\langle v^2\rangle}{v_0}\left(\frac{\omega}{v^*_{L1}}\right)^4\frac{v^*_{L1}}{v_L}\eta^3\alpha^2(1-2\eta^2)h_0(\eta)$$

as well as two transverse waves, one of which is identical to one discussed above, and the other of which propagates through the medium at a velocity $v_{T1}^* = v_T = \sqrt{\mu/\rho}$ and does not decay.

The domain of applicability of the expressions for the averaged elastic characteristics of the microinhomogeneous media obtained by the effective field method has been discussed in a number of papers referred to in /4/. The formulas for the effective elastic moduli of a medium with cracks were discussed in /8, 16/ where they were compared with the experimental data and the exact solutions. In the case of the plane problem the method given good results when the parameter $n_0\pi\langle a^2\rangle$ (a is the half-length of the crack and n_0 is the numerical concentration of the cracks) does not exceed 1.5-2.0. In the three-dimensional case the same procedure was carried out in /7/. Good agreement with experimental data was obtained for values of the parameter $\langle v\rangle/v_0$ of up to 0.66.

Note that the attenuation coefficients are much more sensitive to the details of the spatial distribution of the cracks, than the wave velocity. Indeed, the expressions for γ_L and γ_T in formulas (5.5) and (5.6) contain the integral J of the correlation function $f(x)$ (4.10). As was shown in /5. 10/, the quantity J depends essentially on the spatial distribution of the defects, and a coarse approximation of the function $f(x)$ yields negative coefficients of attenuation even when the concentration of the defects is insignificant. In the case of a low concentration of defects, when the form of the function $f(x)$ is unimportant, the expressions for the attenuation coefficients are identical with the exact expressions obtained in /9—11/.

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